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Existence of Transformed Rational Complex Chebyshev Approximations

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Let X be a compact topological space. Let $C(X)$ be the space of continuous complex functions on X. For g a complex function on X define

$$\|g\| = \sup\{|g(x)| : x \in X\}.$$

Let $\{\phi_1, \ldots, \phi_n\}$, $\{\psi_1, \ldots, \psi_m\}$ be linearly independent subsets of $C(X)$ and define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

Let σ be a continuous mapping of the complex plane into the extended complex plane and define

$$F(A, x) = \sigma(R(A, x)).$$

Let P be a subset of complex $(n+m)$-space. The approximation problem is: Given $f \in C(X)$, to find a parameter $A^* \in P$ for which $e(A) = \|f - F(A, \cdot)\|$ attains its infimum $\rho(f)$ over P. Such a parameter A^* is called best. We study the existence of a best parameter.

A special case of interest is that with $\sigma(x) = x$, that is, approximation by rational functions. Some aspects of this case were recently studied by Dolganov [2], who raised the question of existence. The case of approximation by ratios of power polynomials has been studied by Walsh [5, p. 351].

If $Q(A, x) \neq 0$, $R(\alpha A, x) = R(A, x)$ for all $\alpha \neq 0$. There is therefore no loss of generality in requiring that rational functions $R(A, \cdot)$ be normalized so that

$$\sum_{k=1}^{m} |a_{n+k}| = 1. \tag{1}$$

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Let $\hat{P}$ be the set of all complex coefficient vectors $A = (a_1, ..., a_{n+m})$ satisfying (1).

We need a convention for defining approximations $F(A, \cdot)$ where the denominator $Q(A, \cdot)$ vanishes. We will adapt one due to Boehm [1, 4, p. 84].

DEFINITION. Q has the *dense nonzero property* if for all $Q(A, \cdot) \neq 0$, the set of points at which $Q(A, \cdot)$ does not vanish is dense in X.

If Q has the dense nonzero property we can define $F(A, x)$ if $Q(A, x) = 0$. Let $Q(A, x) = 0$ and define

$$\theta = \limsup_{y \to x} \arg(\sigma(R(A, y)) \qquad Q(A, y) \neq 0,$$

$$r = \limsup_{x \to y} |\sigma(R(A, y))| \qquad Q(A, y) \neq 0, \quad \arg(\sigma(R(A, y))) \to \theta,$$

$$F(A, x) = re^{i\theta}.$$

THEOREM. *Let Q have the dense nonzero property and P be a nonempty closed subset of $\hat{P}$. Let $\sigma(t) \to \infty$ as $t \to \infty$. There exists a best parameter from $\mathscr{P}$ for each $f \in C(X)$.*

Proof. Let $e(A^k)$ be a decreasing sequence with limit $\rho(f) < \infty$. We can assume without loss of generality that $\|f - F(A^1, \cdot)\| < \infty$. If $\|F(A, \cdot) - F(A^1, \cdot)\| > 2\|f - F(A^1, \cdot)\|$ then by the triangle inequality

$$\|f - F(A, \cdot)\| > \|f - F(A^1, \cdot)\|.$$

It follows that $\{\|F(A^k, \cdot)\|\}$ is a bounded sequence and hence $\{\|R(A^k, \cdot)\|\}$ is a bounded sequence. But

$$|R(A, x)| = |P(A, x)|/|Q(A, x)| \geq |P(A, x)| \Big/ \sum_{k=1}^{m} \|\psi_k\|.$$

Hence $\{\|P(A^k, \cdot)\|\}$ is a bounded sequence. It follows by standard arguments [3, p. 25] that the numerator coefficients of $\{A^k\}$ are bounded and the denominator coefficients are bounded by normalization (1). Hence $\{A^k\}$ is a bounded sequence and has an accumulation point A, assume without loss of generality that $\{A^k\} \to A$. If $Q(A, x) \neq 0$, $F(A^k, x) \to F(A, x)$ and

$$|f(x) - F(A, x)| = \lim_{k \to \infty} |f(x) - F(A^k, x)| \leq \rho(f).$$

If $Q(A, x) = 0$

$$|f(x) - F(A, x)| \leq \limsup_{y \to x} |f(y) - \mathscr{F}(A, y)| \leq \rho(f) \qquad Q(A, y) \neq 0.$$

Hence $\|f - F(A, \cdot)\| \leq \rho(f)$.

Examples of Closed Sets of Coefficients

1. $\hat{P}$ is a closed nonempty set.

2. Let K be a closed subset of the complex plane, then

$$P_r = \{A : A \in \hat{P}, Q(A, x) \in K \text{ for all } x \in X\}$$

is a closed subset of $\hat{P}$. A case of particular interest is

$$K = \{z : \mu \leqslant \arg(z) \leqslant \nu\}.$$

3. Let $u \in C(X)$ and

$$P_s = \{A : \operatorname{Re}(F(A, x)) \geqslant \operatorname{Re}(u(x)), x \in X, A \in \hat{P}\}.$$

In the case where Q has the dense nonzero property, P_s is closed.

4. Let $Y = \{y_1, ..., y_s\}$ be a finite subset of X and $w_1, ..., w_s$ be given complex numbers. Let

$$P_t = \{A : A \in \hat{P}, F(A, y_i) = w_i, i = 1, ..., s\}.$$

In the case $m = 1$ (transformed linear approximation) P_t is closed. If $m > 1$, P_t may not be closed and a best approximation may not exist *as shown by the next example.*

Example. Let $X = [0, 1]$ and $F(A, x) = a_1/(a_2 + a_3 x)$. Let $Y = \{y_1\} = \{0\}$ and $w_1 = 1$. Let $f(x) = T_2{}^*(x) = 8x^2 - 8x + 1$. We claim first that for $A \in P_t$, $e(A) > 1$. Suppose this is false then there is $\hat{A}$ such that $e(\hat{A}) \leqslant 1$. If $\hat{a}_1 = 0$, then by Boehm's convention $R(\hat{A}, 0) = 0$, so $\hat{a}_1 \neq 0$. We can, therefore, reparametrize $R(\hat{A}, x)$ as $1/(1 + \alpha x)$. Now if $e(A) \leqslant 1$, then

$$\operatorname{Re}(1/(1 + \alpha/2)) \leqslant 0, \qquad \operatorname{Re}(1/(1 + \alpha)) \geqslant 0;$$

hence

$$\operatorname{Re}(1 + \alpha/2) \leqslant 0, \qquad \operatorname{Re}(1 + \alpha) \geqslant 0,$$

which is impossible. We next observe that if we set

$$A^k = (1/k, 1/k, (k - 1)/k) \text{ then } e(A^k) \to 1.$$

Admissible Approximation

Dolganov defines a rational function $R(A, \cdot)$ to be *admissible* if $\operatorname{Re}(Q(A, \cdot)) > 0$. Even in very simple cases, a best admissible approximation need not exist.

EXAMPLE. Let $X = [-1, -\frac{1}{2}] \cap [\frac{1}{2}, 1]$ and $F(A, x) = a_1/(a_2 + a_3 x)$. Let $f(x) = 1/(ix)$ then

$$|(ix)^{-1} - (k^{-1} + ((k-1)/k)\, ix)^{-1}| \to 0 \text{ uniformly on } X,$$

and $\rho(f) = 0$. There is no admissible $F(A, \cdot)$ with $\|f - F(A, \cdot)\| = 0$.

Even when X is a real interval and F is a ratio of power polynomials, a best admissible approximation need not exist.

EXAMPLE. Let $X = [0, 1]$ and $F(A, x) = a_1/(a_2 + a_3 x + a_4 x^2)$.
Let

$$f(x) = [x(1-x) + i(1 - 2x^2)]^{-1}$$

then

$$|f(x) - [k^{-1} + x(1-x) + i(1 - 2x^2)]^{-1}| \to 0$$

uniformly on X, and $\rho(f) = 0$. But since the denominator of f is i at 0 and $-i$ at 1, f is not expressible as an admissible rational.

REAL APPROXIMATION

Consider the case in which all basis functions are real, all coefficients are real, σ is a continuous mapping of the real line into the extended real line, and f is real. This is the case of real Chebyshev approximation by transformed rational functions. A special case is where $\sigma(x) = x$, which has already been studied by Boehm [1]. The existence theorem obtained earlier in this paper applies. Some cases of closed parameter spaces P are given in [6].

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